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# ON CURVES IN THE LIGHTLIKE CONE

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Abstract. In this paper we define support function for curves with constant cone curvature $\kappa$ in the 2-dimensional lightlike cone and the evolute-involute curves and then characterize curves which satisfy eigenvalue equations for the support function in relation to the evolute-involute curves.

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## 1. Introduction

Lorentzian geometry plays an important role in that transition between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. The fact that relativity theory is expressed in terms of Lorentzian geometry is lucky for geometers, who can thus penetrate surprisingly quickly in to cosmology (redshift, expanding universe and big bang) and a topic no less interesting geometrically, the gravitation of a single star (perihelion procession, bending of light and black holes) [10].

In general relativity, null submanifolds usually appear to be some smooth parts of the achronal boundaries, for example, event horizons of the Kruskal and Kerr black holes and the compact Cauchy horizons in Taub-NUT spacetime and their properties are manifested in the proofs of several theorems concerning black holes and singularities. Degenerate submanifolds of Lorentzian manifolds may be useful to study the intrinsic structure of manifolds with degenerate metric and to have a better understanding of the relation between the existence of the null submanifolds and the spacetime metric [2, 3, 5, 9].

It is important to study submanifolds of the lightlike cone, because of relations between the conformal transformation group and the Lorentzian group of the n-dimensional Minkowski space $E_1^n$ and the submanifolds of the (n+1)-dimensional lightlike cone $Q^{n+1}$[7, 12].

The set of all null vectors in $T_pM$ is called the lightlike cone (or null cone) at $p \in M$, where $M$ is a semi-Riemannian manifold. This terminology derives from relativity theory and particularly in the Lorentzian case, null vectors are also said to be lightlike. For the study on lightlike cone, we refer to [4, 6].

The studies on curves and its evolutes-involutes which satisfy the eigenvalue equations have been done by many mathematicians. For example, in [8, 11], the authors investigated some characterizations of plane curves in term of certain Euclidean curvature properties and stated the classification relating eigenvalues to the geometry of curves. Furthermore, in [1], the authors carried out some results which were given in [8] to nonnull curves in Minkowski plane.

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In this manuscript, due to importance of eigenvalue equation and curves theory in application of the submanifolds theory, we studied the curves and its evolutes-involutes, and characterized them with respect to eigenvalue equations for its support function.

## 2. Curves in the lightlike cone $Q^{n+1}$

Let $E_q^m$ be the $m-$dimensional pseudo-Euclidean space with the metric

$$\overline{G}(x,y) =< x,y >= \sum_{i=1}^{m-q} x_i y_i - \sum_{j=m-q+1}^{m} x_j y_j,$$

where $x = (x_1, x_2, ..., x_m)$, $y = (y_1, y_2, ..., y_m) \in E_q^m$. $E_q^m$ is a flat pseudo-Riemannian manifold of signature $(m-q, q)$.

Let $M$ be a submanifold of $E_q^m$. If the pseudo-Riemannian metric $\overline{G}$ of $E_q^m$ induces a pseudo-Riemannian metric $G$ (respectively, a Riemannian metric, a degenerate quadratic form) on $M$, then $M$ is called a timelike (respectively, spacelike, degenerate) submnifold of $E_q^m$.

Let $c$ be a fixed point in $E_q^m$ and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$S_q^n(c,r) = \{x \in E_q^{n+1} : \overline{G}(x-c, x-c) = r^2\};$$

the pseudo-Riemannian hyperbolic space is define by

$$H_q^n(c,r) = \{x \in E_{q+1}^{n+1} : \overline{G}(x-c, x-c) = -r^2\};$$

the pseudo-Riemannian lightlike cone (quadric cone) is defined by

$$Q_q^n(c) = \{x \in E_q^{n+1} : \overline{G}(x-c, x-c) = 0\}.$$

It is well-known that $S_q^n(c,r)$ is a complete pseudo-Riemannian hypersurface of signature $(n-q, q)$, $q \geq 1$, in $E_q^{n+1}$ with constant sectional curvature $r^{-2}$; $H_q^n(c,r)$ is complete pseudo-Riemannian hypersurface of signature $(n-q, q)$, $q \geq 1$, in $E_{q+1}^{n+1}$ with constant sectional curvature $-r^{-2}$; $Q_q^n(c)$ is a degenerate hypersurface in $E_q^{n+1}$. The spaces $E_q^n$, $S_q^n(c,r)$ and $H_q^n(c,r)$ are called pseudo-Riemannian space forms. The point $c$ is called the center of $S_q^n(c,r)$, $H_q^n(c,r)$ and $Q_q^n(c)$. When $c = 0$ and $q = 1$, we simply denote $Q_1^n(0)$ by $Q^n$ and call it the lightlike cone (or simply the light cone) [10].

Let $E_1^{n+2}$ be the $(n+2)-$dimensional Minkowski space and $Q^{n+1}$ the lightlike cone in $E_1^{n+2}$. A vector $\alpha \neq 0$ in $E_1^{n+2}$ is called spacelike, timelike or lightlike, if $< \alpha, \alpha >> 0$, $< \alpha, \alpha >< 0$ or $< \alpha, \alpha >= 0$, respectively. A frame field $\{e_1, .., e_n, e_{n+1}, e_{n+2}\}$ on $E_1^{n+2}$ is called an asymptotic orthonormal frame field, if

$$< e_{n+1}, e_{n+1} >=< e_{n+2}, e_{n+2} >= 0, \quad < e_{n+1}, e_{n+2} >= 1,$$
$$< e_{n+1}, e_i >=< e_{n+2}, e_i >= 0, \quad < e_i, e_j >= \delta_{ij}, \quad i,j = 1, ..., n.$$

We assume that the curve

$$x : I \to Q^{n+1} \subset E_1^{n+2}, \quad t \to x(t) \in Q^{n+1}, \quad t \in I \subset R$$

is a regular curve in $Q^{n+1}$. In the following, we always assume that the curve regular and $x(t) \nparallel x'(t) = \frac{dx(t)}{dt}$, for all $t \in I$.

**Definition 2.1.** *A curve $x(t)$ in $E_1^{n+2}$ is called a Frenet curve, if for all $t \in I$, the vector fields $x(t)$, $x'(t)$, $x''(t), ..., x^{(n)}(t)$, $x^{(n+1)}(t)$ are linearly independent and the vector fields $x(t)$, $x'(t)$, $x''(t), ..., x^{(n+1)}(t)$, $x^{(n+2)}(t)$ are linearly dependent, where $x^{(n)}(t) = \frac{dx^n(t)}{dt^n}$. Since $< x, x >= 0$*

*and* $< x, dx >= 0$, $dx(t)$ *is spacelike. Then the induced arc length (or simply the arc length)* $s$ *of the curve* $x(t)$ *can be defined by*

$$ds^2 =< dx(t), dx(t) > .$$

If we take the arc length $s$ of the curve $x(t)$ as the parameter and denote $x(s) = x(t(s))$, then $x'(s) = \frac{dx}{ds}$ is a spacelike unit tangent vector field of the curve $x(s)$. Now we choose the vector field $y(s)$, the spacelike normal space $V^{n-1}$ of the curve $x(s)$ such that they satisfy the following conditions:

$$< x(s), y(s) >= 1, \quad < x(s), x(s) >=< y(s), y(s) >=< x'(s), y(s) >= 0,$$
$$V^{n-1} = \{span_R\{x, y, x'\}\}^{\perp}, \quad span_R\{x, y, x', V^{n-1}\} = E_1^{n+2}.$$

Therefore, choosing the vector fields $\alpha_2(s), \alpha_3(s), ..., \alpha_n(s) \in V^{n-1}$ suitably, we have the following Frenet formulas

$$\begin{aligned}
&x'(s) = \alpha_1(s)\\
&\alpha_1'(s) = \kappa_1(s)x(s) - y(s) + \tau_1(s)\alpha_2(s)\\
&\alpha_2'(s) = \kappa_2(s)x(s) - \tau_1(s)\alpha_1(s) + \tau_2(s)\alpha_3(s)\\
&\alpha_3'(s) = \kappa_3(s)x(s) - \tau_2(s)\alpha_2(s) + \tau_3(s)\alpha_4(s)\\
&\qquad\qquad ...\\
&\alpha_i'(s) = \kappa_i(s)x(s) - \tau_{i-1}(s)\alpha_{i-1}(s) + \tau_i(s)\alpha_{i+1}(s)\\
&\qquad\qquad ...\\
&\alpha_{n-1}'(s) = \kappa_{n-1}(s)x(s) - \tau_{n-2}(s)\alpha_{n-2}(s) + \tau_{n-1}(s)\alpha_n(s)\\
&\alpha_n'(s) = \kappa_n(s)x(s) - \tau_{n-1}(s)\alpha_{n-1}(s)\\
&y'(s) = -\sum_{i=1}^{n} \kappa_i(s)\alpha_i(s),
\end{aligned} \tag{1}$$

where $\alpha_2(s), \alpha_3(s), ..., \alpha_n(s) \in V^{n-1}$, $< \alpha_i, \alpha_j >= \delta_{ij}$, $i, j = 1, 2, ..., n$. The function $\kappa_1(s), ...,$ $\kappa_n(s), \tau_1(s), ..., \tau_{n-1}(s)$ are called cone curvature functions of the curve $x(s)$. The frame

$$\{x(s), y(s), \alpha_1(s), \alpha_2(s), ..., \alpha_n(s)\}$$

is called the asymptotic orthonormal frame on $E_1^{n+2}$ along the curve $x(s)$ in $Q^{n+1}$.

## 3. Curves with constant cone curvature in the lightlike cone $Q^2$

In this section, we consider curves in the lightlike cone $Q^2$ and define support function for curves with constant cone curvature $\kappa$ in the $2-$dimensional lightlike cone and the evolute-involute curves and then characterize curves which satisfy eigenvalue equations for the support function in relation to the evolute-involute curves.

Let $x : I \to Q^2 \subset E_1^3$ be a curve, then from (1), we have

$$\begin{aligned}
&x'(s) = \alpha(s),\\
&y'(s) = -\kappa(s)\alpha(s),\\
&\alpha'(s) = \kappa(s)x(s) - y(s),
\end{aligned} \tag{2}$$

where $s$ is an arc length parameter of the curve $x(s)$ and $x(s)$, $y(s)$, $\alpha(s)$ satisfy

$$< x, x >=< y, y >=< x, \alpha >=< y, \alpha >= 0,$$
$$< x, y >=< \alpha, \alpha >= 1.$$

For an arbitrary parameter $t$ of the curve $x(t)$, the cone curvature function $\kappa$ is given by

$$\kappa(t) = \frac{< \frac{dx}{dt}, \frac{d^2x}{dt^2} >^2 - < \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2} >< \frac{dx}{dt}, \frac{dx}{dt} >}{2 < \frac{dx}{dt}, \frac{dx}{dt} >^5}. \tag{3}$$

Using a orthonormal frame on the curve $x(s)$ and denoting by $\overline{\kappa}$, $\overline{\tau}$, $\beta$ and $\gamma$ the curvature, the torsion, the principal normal and the binormal of the curve $x(s)$ in $E_1^3$, we have

$$\begin{aligned} x' &= \alpha \\ \alpha' &= \kappa x - y = \overline{\kappa}\beta, \end{aligned}$$

where $\kappa \neq 0$, $< \beta, \beta >= \varepsilon = \pm 1$, $< \alpha, \beta >= 0, < \alpha, \alpha >= 1$, $\varepsilon\kappa < 0$. Then we get

$$\beta = \varepsilon \frac{\kappa x - y}{\sqrt{-2\varepsilon\kappa}}, \quad \varepsilon\overline{\tau}\gamma = \frac{\kappa'}{2\sqrt{-2\varepsilon\kappa}}(x + \frac{1}{\kappa}y). \tag{4}$$

Choosing

$$\gamma = \sqrt{\frac{-\varepsilon\kappa}{2}}(x + \frac{1}{\kappa}y) \tag{5}$$

we obtain

$$\overline{\kappa} = \sqrt{-2\varepsilon\kappa}, \quad \overline{\tau} = -\frac{1}{2}(\frac{\kappa'}{\kappa}). \tag{6}$$

**Theorem 3.1.** *The curve $x : I \to Q^2$ is a planar curve if and only if the cone curvature function $\kappa$ of the curve $x(s)$ is constant [6].*

If the curve $x : I \to Q^2 \subset E_1^3$ is a planar cure, then we have the following Frenet formulas

$$\begin{aligned} &x' = \alpha, \\ &\alpha' = \varepsilon\sqrt{-2\varepsilon\kappa}\beta, \\ &\beta' = -\sqrt{-2\varepsilon\kappa}\alpha. \end{aligned} \tag{7}$$

**Definition 3.1.** *Let $x : I \to Q^2$ be a curve with constant cone curvature $\kappa$ and arc length parameter $s$, then the support function of $x(s)$ with respect to a fixed point $p_0 \in Q^2$ is defined by*

$$\rho(p_0) =< \beta, p_0 - x > . \tag{8}$$

Differentiating (8) with respect to $s$ and using (7), we have

$$\rho'(p_0) = -\sqrt{-2\varepsilon\kappa} < \alpha, p_0 - x > \tag{9}$$

and we get a representation of $x(s)$ in terms of the support function:

$$x - p_0 = \frac{1}{\sqrt{-2\varepsilon\kappa}}\rho'(p_0)\alpha - \varepsilon\rho(p_0)\beta. \tag{10}$$

**Theorem 3.2.** *Let $x : I \to Q^2$ be a curve with constant cone curvature $\kappa$ and arc length parameter $s$, then the support function $\rho(p_0)$ of $x(s)$ with respect to a fixed point $p_0 \in Q^2$ can be written as follows:*

*(i)* $\rho(p_0)(s) = c_1 s + c_2$,

*for $\kappa = 0$, it is a part of circle;*

*(ii)* $\rho(p_0)(s) = c_1 \cos\sqrt{-2\kappa}s + \sin\sqrt{-2\kappa}s + \frac{1}{\sqrt{-2\kappa}}$,

*for $\varepsilon = 1$, $\kappa < 0$, it is an ellipse;*

*(iii)* $\rho(p_0)(s) = c_1 \cosh\sqrt{2\kappa}s + c_2 \sinh\sqrt{2\kappa}s - \frac{1}{\sqrt{2\kappa}}$,

*for $\varepsilon = -1$, $\kappa > 0$, it is a hyperbola; where $c_1, c_2 \in E_1^3$.*

*Proof.* Differentiating (10), we get

$$\frac{1}{\sqrt{-2\varepsilon\kappa}}\rho''(p_0) + \varepsilon\sqrt{-2\varepsilon\kappa}\rho(p_0) = 1. \tag{11}$$

Solving this equation (11), we obtain that

$$\rho(p_0)(s) = c_1 s + c_2, \quad \kappa = 0,$$
$$\rho(p_0)(s) = \rho(p_0)(s) = c_1 \cos\sqrt{-2\kappa}s + \sin\sqrt{-2\kappa}s + \frac{1}{\sqrt{-2\kappa}}, \quad \varepsilon = 1,\ \kappa < 0,$$
$$\rho(p_0)(s) = c_1 \cosh\sqrt{2\kappa}s + c_2 \sinh\sqrt{2\kappa}s - \frac{1}{\sqrt{2\kappa}}, \quad \varepsilon = -1, \kappa > 0,$$

where $c_1, c_2 \in E_1^3$. □

**Definition 3.2.** *Let $x : I \to Q^2$ be a curve with constant cone curvature $\kappa$ and arc length paraameter $s$. Then the locus of the centre of curvature of a planar curve $x(s)$ is called the evolute of the curve $x(s)$ and given by*

$$E_x(s) = x(s) + \frac{1}{\sqrt{-2\varepsilon\kappa}}\beta(s). \tag{12}$$

From (12), the evolute curve $E_x$ is not regular curve.

**Definition 3.3.** *Let $x : I \to Q^2$ be a curve with constant cone curvature $\kappa$ and arc length parameter $s$. For a fixed value $s_1 \in \mathbb{R}$, the involute of the curve $x(s)$ is defined by*

$$\mathcal{I}_{x,s_1}(s) := x(s) - (s + s_1)\alpha(s). \tag{13}$$

If we derivative of (13) with respect to $s$, we get

$$\mathcal{I}'_{x,s_1}(s) = -\varepsilon(s + s_1)\sqrt{-2\varepsilon\kappa}\beta(s). \tag{14}$$

Thus the condition $(s + s_1) \neq 0$ is equivalent to the regularity of the involute $\mathcal{I}_{x,s_1}$ and we suppose all involutes to be regular. If $s_1$ varies, one obtains a one-parameter family of involutes.

If $\{\alpha, \beta\}$ is an orthonormal frame of $x$ then

$$\alpha_I = -\varepsilon sign(s + s_1)\beta$$
$$\beta_I = \varepsilon sign(s + s_1)\alpha \tag{15}$$

defines an orthonormal frame of the involute $\mathcal{I}_{x,s_1}$, where $< \alpha_I, \alpha_I >= \varepsilon$ and $< \beta_I, \beta_I >= 1$.

**Theorem 3.3.** *Let $x : I \to Q^2$ be a curve with constant cone curvature $\kappa$ and arc length parameter $s$. Then the involute $\mathcal{I}_{x,s_1}$ of $x(s)$ satisfies the following properties:*

*(i) If $\{\alpha, \beta\}$ is an orthonormal frame of $x(s)$ then*

$$\{\alpha_I = -\varepsilon sign(s + s_1)\beta,\ \beta_I = \varepsilon sign(s + s_1)\alpha\}$$

*is an orthonoral frame of $\mathcal{I}_{x,s_1}$.*

*(ii) The cone curvature function $\kappa_I$ of the involute curve $\mathcal{I}_{x,s_1}$ satisfies:*

$$\kappa_I = -\frac{\varepsilon}{2(s + s_1)^2} \tag{16}$$

*and we insert the equation (13)*

$$\mathcal{I}_{x,s_1}(s) := x(s) - sign(s + s_1)\sqrt{\frac{-\varepsilon}{2\kappa_I}}\alpha(s).$$

*Proof.* Considering (7) and (15), we have

$$\sqrt{-2\varepsilon\kappa_I} =< \alpha'_I, \beta_I >= -\sqrt{-2\varepsilon\kappa},$$

that means that $\kappa = \kappa_I$.

From (14), we get

$$(s+s_1)^2 = -\frac{\varepsilon}{2\kappa_I}$$

and

$$s+s_1 = sign(s+s_1)\sqrt{\frac{-\varepsilon}{2\kappa_I}}. \tag{17}$$

Thus we can write (3.12) as follow:

$$\mathcal{I}_{x,s_1}(s) := x(s) - sign(s+s_1)\sqrt{\frac{-\varepsilon}{2\kappa_I}}\alpha(s).$$

□

**Theorem 3.4.** *Let $x : I \to Q^2$ be a curve with constant cone curvature $\kappa$ and arc length parameter $s$. If there exists an appropriate $s_1 \epsilon \mathbb{R}$ such that the evolute curve*

$$E_x(s) = x(s) + \frac{1}{\sqrt{-2\varepsilon\kappa}}\beta(s)$$

*of $x(s)$ and the involute curve $\mathcal{I}_{x,s_1}(s)$ of $x(s)$ associated to $s_1$,*

$$\mathcal{I}_{x,s_1}(s) := x(s) - (s+s_1)\alpha(s)$$

*satisfy the relation*

$$E_x = -\lambda\mathcal{I}_{x,s_1} + const.,$$

*then there exists $p_0 \epsilon Q^2$ such that the support function $\rho(p_0)$ with respect to $p_0$ satisfies the homogeneous eigenvalue equation*

$$\rho''(p_0) + 2\kappa\lambda\rho(p_0) = 0.$$

*Proof.* If we use the equations (12) and (13), we have

$$E_x + \lambda\mathcal{I}_{x,s_1} = (1+\lambda)x - (s+s_1)\lambda\alpha + \frac{1}{\sqrt{-2\varepsilon\kappa}}\beta$$

and using (7) we get

$$(E_x + \lambda\mathcal{I}_{x,s_1})' = -\lambda(s+s_1)\varepsilon\sqrt{-2\varepsilon\kappa}\beta.$$

Thus, there exists an apropriate $s_1 \epsilon \mathbb{R}$ such that

$$E_x + \lambda\mathcal{I}_{x,s_1} = const. \tag{18}$$

Let $E_x + \lambda\mathcal{I}_{x,s_1} = const. = (1+\lambda)p_0$ for an appropriate $p_0 \epsilon Q^2$. Then we have

$$\lambda(\mathcal{I}_{x,s_1} - p_0) = -E_x + p_0; \tag{19}$$

that means that $E_x$ and $\mathcal{I}_{x,s_1}$ are homothetic (without translation).

If we consider (10) and (19), we get

$$\left[\left(\lambda\frac{1}{\sqrt{-2\varepsilon\kappa}} + \frac{1}{\sqrt{-2\varepsilon\kappa}}\right)\rho' - \lambda(s+s_1)\right]\alpha + \left[-\varepsilon(1+\lambda)\rho + \frac{1}{\sqrt{-2\varepsilon\kappa}}\right]\beta = 0.$$

The coefficients of the frame $\{\alpha, \beta\}$ must vanish, this means that

$$\rho(p_0) = \frac{\varepsilon}{(1+\lambda)\sqrt{-2\varepsilon\kappa}} \tag{20}$$

Considering (11) into (20), we have

$$\rho''(p_0) + 2\kappa\lambda\rho(p_0) = 0. \tag{21}$$

□

Now we determine the curves with constant cone curvature $\kappa$ in 2-dimensional lightlike cone $Q^2$ which satisfy the homogeneous eigenvalue equation related with its evolutes and involutes (21):

*Case 1.* Let $\lambda = 0$.

In this case $\rho''(p_0) = 0$ and a curve satisfying this equation is part of a circle.

*Case 2.* Let $\lambda \neq -1$ and $\lambda\kappa < 0$.

In this case, we can write the general solution of the homogeneous eigenvalue equation (21) in the form

$$\rho(p_0)(s) = c_1 \cosh \sqrt{-2\lambda\kappa}s + c_2 \sinh \sqrt{-2\lambda\kappa}s.$$

*Case 3.* Let $\lambda \neq -1$ and $\lambda\kappa > 0$.

In this case, we can write the general solution of the homogeneous eigenvalue equation (21) in the form

$$\rho(p_0)(s) = \cos c_1 \sqrt{2\lambda\kappa}s + c_2 \sin \sqrt{2\lambda\kappa}s.$$

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